AIAA 79-1492R

Propagation of Linear Wave Packets in Laminar Boundary Layers

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A wave packet generated by a localized impulse can be defined in terms of an integral or sum of periodic wavetrains. This integral can be represented asymptotically in a number of different ways. Wave forms of a two-dimensional wave packet defined by various expansions are evaluated numerically and compared over a range of distances from their point of origin.

I. Introduction

A PROPER understanding of how wave packets form in unstable flows and how they propagate downstream and eventually evolve into a turbulent flow can contribute not only to our understanding of many fundamental processes that take place during transition, but will also provide guidance on the more urgent problem of predicting transition.

In the context of aeronautics we are concerned with the situation where transition occurs through the selective amplification of weak vorticity perturbations that are generated by various external sources, such as freestream turbulence, boundary vibration, or sound emanating from engines. Once these disturbances have been created they amplify and propagate downstream. Initially, when the amplitudes of these fluctuations are sufficiently weak, and the nonlinear stress terms are negligible compared with those driving the base flow, their progress is governed by the linearized form of the Navier-Stokes equations. Theory based on these linearized equations provides a reasonable basis for calculating the amplification of these instability waves. Although the excitation process by which external disturbances create waves ("receptivity," as Morkovin has called it) is understood to a degree, it is still not possible to compute the strengths of the waves created by a given flow environment (such as a known freestream turbulence). Nevertheless, because the linear growth can be very large, it has been possible largely to ignore this aspect of the overall transition process and still come up with a reasonable prediction of the eventual transition position, by using the "e^9" method proposed by A.M.O. Smith and by Van Ingen many years ago. In this approach, the degree of amplification from the initial point of instability is computed for every wave, and transition is deemed to occur when this factor has reached a sufficiently large quantity, namely e^9, for the most unstable wave. In order to make a prediction of the position of transition in a given flow by this method one needs to be able to compute the amplification of the most unstable wave according to linear theory. In simple two-dimensional flows this is not too difficult to accomplish and, with certain reservations, can provide a reasonably useful estimate of the position of transition to turbulence. But when the same techniques are applied to more complex situations involving, for example, either three-dimensional or growing boundary layers, there are a number of additional factors to be taken into account. First, it is not quite clear how to define the "most unstable wave," because of uncertainty in the direction of its propagation. Second, since some of these flows support highly unstable modes it seems to be necessary to take more account of the actual breakdown mechanism, as the e^9 criterion is inadequate as it stands.

Previously the whole of the prediction approach has beeen centered on the behavior of monochromatic waves. It turns out, however, that if the evolution of a more general type of disturbance involving a broad spectrum of waves is considered, the first of the above difficulties can be resolved. Natural disturbances do, of course, excite a broad spectrum of waves from distributed sources throughout the flow, and this leads to deeply modulated wave trains containing elements of different frequencies and orientations. The excitation process can be thought of as a series of distributed random-point sources that create isolated packets, randomly phased with one another in both space and time. It appears to be a more straightforward process to predict the propagation of one of these isolated packets of waves through a complex three-dimensional spatially evolving flow compared with the prediction of the path of one of the elemental Fourier components.

The study of the evolution of wave packets is also of importance as far as the second point regarding the actual breakdown process is concerned. It has been observed in experiment that modulated wave trains break down to turbulence at lower amplitudes than periodic waves. This has been clearly demonstrated by experiments on artificially excited wave packets, which also exhibit breakdown through secondary and then tertiary modes of instability at surprisingly low intensities of the primary wave.[1] The reason for this lies in the fact that the nonlinear Reynolds stresses induced by the passage of a modulated wave packet are quite different from those created within a periodic wavetrain. In fact, for short packets consisting of only a few cycles, the traveling stress field will be largely dominated by terms in the equations of motion that arise from the modulation, and this factor has somehow to be taken into account if prediction is to be improved.

Since the behavior of wave packets plays such an important role in the transition process, it is vital to develop techniques for predicting their evolution. Ideally one would like to be able to solve the problem of the development of an isolated three-dimensional wave packet as it propagates downstream in a growing boundary layer, following the process into at least the stage of weak nonlinearity. But here we concentrate solely on the linear phase of growth and, in particular, on flows that are planar. At large distances from the source it will be shown that the wave packet approaches a slowly modulated wave train that can be described in terms of fast and slow time scales. The local fast oscillation arises as an almost monotonic traveling wave, scaling with either X or T, while the slow scale appears as the amplitude modulation scaling with X/T. An asymptotic solution can be expected to describe this type of behavior. Not only would this provide a rapid way of predicting wave packet evolution, but it would also give a clean mathematical formulation required for any theoretical extension to cover such things as weak nonlinearity

Presented as Paper 79-1492 at the AIAA 12th Fluid and Plasma Dynamics Conference, Williamsburg, Va., July 23-25, 1979; submitted Sept. 29, 1979; revision received Sept. 8, 1980.

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or the effects of slow spatial growth. There has been considerable confusion in recent years as to the best way to obtain asymptotic solutions of wave packets, and it is hoped that the ensuing discussion and the resulting numerical comparison will clarify some of these ideas.[2,3]

This paper considers the prediction of the development of an isolated wave packet. Various asymptotic representations are discussed and compared numerically with a direct summation for the particular case of the two-dimensional wave packet in a "parallel" mean flow having a Blasius boundary layer velocity profile.

II. Analytical Treatment of Wave Packets

Isolated wave packets can be created by impulsively disturbing the flow at a point. All possible modes supported by the wave system will then be excited, and each of these individual "Fourier" components will amplify or decay independently according to the dispersion relations for the particular flow. Some time after the initiation of the disturbance the overall motion will be formed from the sum of these Fourier modes suitably weighted, in both phase and gain, by factors appropriate to the dispersion law. In the case of a parallel flow (not necessarily two-dimensional) the linearized perturbation equations separate, and the normal modes arise in the form

$$\Phi(y;\alpha,\beta,\omega)\exp i\{\alpha X+\beta Z-\omega T\} \qquad (1)$$

where the wavenumbers α and β are related to the frequency parameter, ω, through a dispersion relation

$$\omega=F(\alpha,\beta) \qquad (2)$$

and $\Phi(y)$ is the internal modal structure of some linear perturbation quantity such as velocity or pressure. The dispersion law provides the link between the eigenvalues of the disturbance equation, in this case the Orr-Sommerfeld equation. An impulsive form of excitation will result in a disturbance formed from a sum of these normal modes. The complete solution at some downstream station requires a knowledge of the input spectrum or receptivity as well as the phase and gain of each element. So far this aspect of the problem has not been treated properly, and perhaps the best assumption that can be made is that all modes are initially equally represented, i.e., that the input spectrum is flat. The resulting packet is then:

$$\sim\iint\Phi(y;\alpha,\beta,\omega)\exp i\{\alpha X+\beta Z-\omega T\}\mathrm{d}\alpha\mathrm{d}\beta\ldots \qquad (3)$$

Since this integral will in any case be dominated by exponentially large elements, the neglect of any algebraic weighting function from the input spectrum is not likely to be very significant. Similarly, the function defining the internal structure of the eigensolutions will also only provide a weak modification to the exponential terms, and in this paper will be ignored. In practical terms, even in fairly simple examples, the computations involved in evaluating Eq. (3) in order to obtain the disturbance by direct summation can be excessive. It is natural, therefore, to explore other methods of estimating this integral. Asymptotic methods that are valid for large X or T have particular appeal, as it then seems possible to exploit the "ray-like" propagation character of the waves, with the prospect of extending the analysis to include slow spatial variations in the dispersions, or even the effects of weak nonlinearity.

Wave trains occur in a great many physical phenomena, and the concept of an impulsively generated wave packet is common. In most of the cases discussed in the literature[4] the wave systems are conservative. Then every spectral component that is excited by a source is bound to propagate outward from that source with its appropriate group velocity. Along each ray, defined by the group velocity, the motion will be dominated by waves of a specific frequency and wavenumber, the magnitudes of these waves being controlled both by interference with waves originating from neighboring parts of the spectrum and by the degree of input excitation at that frequency. At very large distances from the source the disturbance develops into a form containing both fast and slow scales. Then the ray behavior is particularly evident and the concept of different wavenumber disturbances propagating outward from the source at the appropriate group velocities is supported by the asymptotic representation obtained by the method of stationary phase. In problems involving an instability the wave system is no longer conservative, and some care is required in deriving the proper asymptotic form of solution. But because it is often found that the instability mechanism is extremely weak, and that the dispersion relations therefore contain only a relatively small imaginary component, it is natural to attempt to try and extend the ray methods, which have proved so powerful in conservative systems, to these conditions. This often is done by using the real part of the dispersion to define the trajectories of the various wave groups, and then by applying some weighting to the amplitudes in order to take account of the degree of amplification or damping appropriate to each particular band of waves. This idea has some appeal and seems reasonable from a physical viewpoint, where it might well be expected that the phase-cancelling process must still be largely responsible for the creation of packet envelopes. This approach is not supported by careful analysis in all cases[5] and a more direct approach is needed.

A. Steepest Descent

The asymptotic representation of an integral of the form Eq. (3) can be found by the method of steepest descent. The author has discussed this approach[6-8] and obtained the general result for the leading term along the rays X/T and Z/T in the form

$$\sim\frac{\exp i\{\alpha^*X+\beta^*Z-\omega(\alpha^*,\beta^*)T\}}{T\left[\dfrac{\partial^2\omega}{\partial\alpha^2}\dfrac{\partial^2\omega}{\partial\beta^2}-\left(\dfrac{\partial^2\omega}{\partial\alpha\partial\beta}\right)^2\right]} \qquad (4)$$

where the quantities α^* and β^* were defined by the relations

$$\frac{X}{T}=\frac{\partial\omega}{\partial\alpha}(\alpha^*,\beta^*)\text{ and }\frac{Z}{T}=\frac{\partial\omega}{\partial\beta}(\alpha^*,\beta^*) \qquad (5)$$

In evaluating the double integral the contours in the two planes are distorted so that they pass through saddlepoints so as to eliminate the rapid variations of the oscillations in the exponent and to maximize its real component. There is, of course, nothing new in this treatment; the result has been shown many times before. The difficulty with the result is that the dispersion relation is generally only known for real frequencies and this information is not directly useful in the evaluation of the above expressions for real rays in the X/T and Z/T planes, except where the amplification (with respect to time) is at a maximum, and $\partial\omega/\partial\alpha$ and $\partial\omega/\partial\beta$ are then always real.[6] Since we are interested in cases where both X/T and Z/T are real, the quantities $\partial\omega/\partial\alpha$ and $\partial\omega/\partial\beta$ must also be real, and can readily be identified with the group velocities in the X/T and Z/T directions, although relating to waves that may have complex wavenumbers and a complex frequency parameter. This result was originally derived for two-dimensional flows, but it does, of course, apply to all planar flows. This asymptotic expansion recently has been applied to the special case where ω is a fixed real parameter, and the result then defines the wedge of disturbance generated by an oscillating point source. Then each ray Z/X is defined by purely real $\partial\alpha/\partial\beta$ and provides the necessary link between local values of the starred variables α^*, β^* for each ω. The same method has been applied to certain problems involving three-dimensional boundary layers in general,[10] and in particular for tracing rays in boundary layer flows on a rotating disk.[9]

For the specific case of a two-dimensional packet Eq. (4) reduces to

$$\sqrt{\frac{2\pi}{(\partial^2\omega/\partial\alpha^2)(\alpha^*)T}}\times\exp i\{\alpha^*X-\omega(\alpha^*)T\}$$

$$\times\{1+O(1/T)+\ldots\} \tag{6}$$

where

$$\frac{\partial\omega}{\partial\alpha}(\alpha^*)=\frac{X}{T}$$

and again the evaluation of Eq. (6) over a range of X/T values requires a knowledge of the complex dispersion relations $\omega(\alpha)$.

B. The Gaussian Model

The full asymptotic solution requires the ability to determine the eigenvalues and the first and second derivatives at points on the complex wavenumber plane. This is not always convenient, as it generally requires a great deal of computation, and so approximations to the dispersion are sought. Far downstream, the solution is dominated by waves with the greatest amplification rates, that is, those associated with the ray having ω_i as a maximum. This point lies on the real α_r-axis where $\partial\omega_i/\partial\alpha_r$ is zero. By expanding the dispersion relation about this point and truncating after the second derivative the integral can be evaluated completely in the form[11,12]

$$\sqrt{\frac{2\pi}{(\partial^2\omega/\partial\alpha_m^2)T}}\times\exp i\Big\{\Big(\alpha_m\frac{X}{T}-\omega_m\Big)$$

$$-\Big(\frac{X}{T}-\frac{\partial\omega}{\partial\alpha}m\Big)^2\Big/2\frac{\partial^2\omega}{\partial\alpha^2}m\Big\}T \tag{7}$$

where m denotes values at the maximum amplification ($\partial\omega_i/\partial\alpha=0$). A similar expression can also be derived for an expansion about the most unstable mode in the spatial sense. This will occur about a point where $\omega_i=0$, and in the case of boundary layer instability will be close to that defining the most unstable temporal mode.

C. The Real Axis Approximation

When the instability is weak the dispersion can be considered to be almost a real function, and for the purposes of calculating the asymptotic behavior it is often assumed that the imaginary component can be ignored in determining the ray trajectories. This does not seem to apply to the dispersion relation arising in boundary layer instability waves. When the dispersion relation is known completely along the real axis α_r, say, analytic continuation can be used to define ω in the complex domain, and can be used to evaluate the saddlepoint position and the necessary parameters there for the asymptotic formulation. Expanding the dispersion relations about a point on the real axis ($\alpha_i=0$), denoted by Eq. (1), say, expressions for ω and its derivatives at a saddle-point can be obtained.

$$\omega^*=\omega(1)+(\alpha^*-\alpha(1))\frac{\partial\omega}{\partial\alpha}(1)$$

$$+\frac{(\alpha^*-\alpha(1))^2}{2!}\frac{\partial^2\omega}{\partial\alpha^2}(1)+\ldots \tag{8}$$

and

$$\frac{\partial\omega}{\partial\alpha}(\alpha^*)=\frac{\partial\omega}{\partial\alpha}(1)+(\alpha^*-\alpha(1))\frac{\partial^2\omega}{\partial\alpha^2}(1)+\ldots \tag{9}$$

Since $(\partial\omega/\partial\alpha)(\alpha^*)=X/T$ we have

$$\alpha^*-\alpha(1)\simeq\Big[\frac{X}{T}-\frac{\partial\omega}{\partial\alpha}(1)\Big]\Big/\frac{\partial^2\omega}{\partial\alpha^2}(1)$$

for small $\alpha^*-\alpha(1)$ [i.e., when the saddle point lies close to Eq. (1)]. If Eq. (1) is chosen so that $(\partial\omega/\partial\alpha)(1)=X/T$ (and $\alpha_i=0$), we get

$$\sqrt{\frac{2\pi}{(\partial^2\omega/\partial\alpha^2)(1)T}}\times\exp i\Big\{\Big(\alpha(1)\frac{X}{T}-\omega(1)\Big)$$

$$+\Big(\frac{\partial\omega_i}{\partial\alpha}(1)\Big)^2\Big/2\frac{\partial^2\omega}{\partial\alpha^2}(1)\Big\}T\ldots \tag{10}$$

and if the term $(\partial\omega_i/\partial\alpha)^2/\partial^2\omega/\partial\alpha^2$ is sufficiently small

$$\sqrt{\frac{2\pi}{(\partial^2\omega/\partial\alpha^2(1)T}}\times\exp i\{\alpha(1)X-\omega(1)T\} \tag{11}$$

Thus provided the factor $(\partial\omega_i/\partial\alpha)^2/(\partial^2\omega/\partial\alpha^2)$ is small enough the solution takes on the form given by a simple extension of the result for conservative wave systems.[2]

III. Numerical Solutions

Since the solution of the Orr-Sommerfeld equation for the evaluation of even one specific eigenvalue is time consuming, the amount of computation required to find the dispersion over the complex plane so as to cope with either direct summation or the steepest descent is generally prohibitive.

The dispersion relation can be defined in limited regions of the α and β planes by a series of the form

$$\frac{\omega}{\alpha}=\sum_{n=0}^{N}\sum_{m=0}^{M}A_{nm}(\alpha R-(\alpha R)_0)^n(\alpha^2+\beta^2-\alpha_0^2)^m \tag{12}$$

The coefficients A_{nm} have been evaluated for M, N up to 20 for the Blasius profile by a contour integration technique,[13] and Eq. (12) can be used to evaluate the various expressions for wave packets that have been derived. It turns out that when $(\alpha R-(\alpha R)_0)$ and $(\alpha^2+\beta^2-\alpha_0^2)$ are not very small, the series is insufficiently convergent to give accurate values of ω, but this difficulty can be overcome by applying a nonlinear Shanks transformation to the partial sums in order to improve the rate of convergence. Then eigenvalues can be found to an accuracy of six decimal points within the region of the neutral loop. It is worth noting that whereas a direct numerical attack on these models would involve a large numbers of solutions of the Orr-Sommerfeld equation, and thus constitutes a formidable task even on a mainframe computer, evaluation by the series can be carried out on a desktop computer. In fact, the following numerical comparisons were all performed on a Commodore PET microcomputer. In the following computations, the Reynolds number R was taken to be 1000.

A. Summation

The integral defining a two-dimensional packet is:

$$I=\int\exp i\{\alpha X-\omega T\}\,\mathrm{d}\alpha$$

This was evaluated numerically for different values of X and T by writing the integral as a sum over α in the range 0.08 to 0.4 in 64 steps. The values of ω were calculated by the power

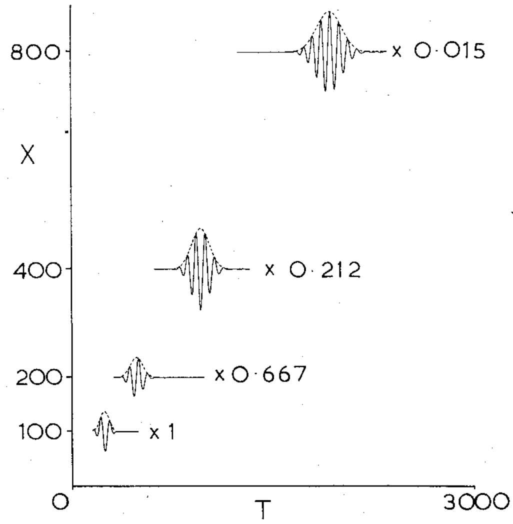

Fig. 1 Wave packets formed by summation.

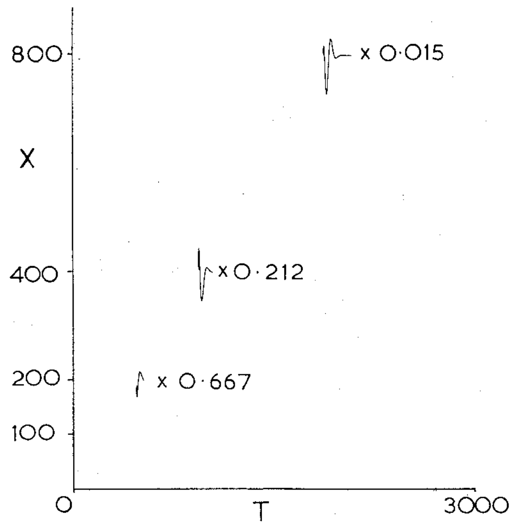

Fig. 3 Real axis approximation.

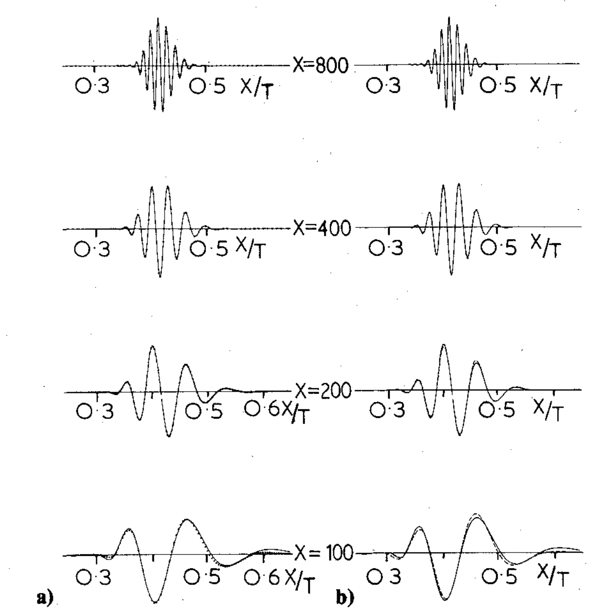

Fig. 2 Comparison with summation solution. a) Method of steepest descent. b) Gaussian approximtion.

series representation of the dispersion using 14 terms. The results of the summation at different distances from the source are displayed on Fig. 1, where the real component shows the perturbation signal as a function of the time from the initiation of the disturbance, and the modulus shows the envelope. At large times (large X and small X/T) the wave packet exhibits some ringing in the tails of the signal. This arises because the integral defining the solution has been approximated by a finite sum with insufficient resolution. In fact, this demonstrates one of the problems in attempting to compute wave packets at large distances from the source by direct evaluation of the Fourier integral, and explains why the amount of computing can become prohibitive. For large values of X or T the distribution approaches a slowly modulated wave train, and it would seem more appropriate, therefore, to use some form of asymptotic, or ray theory, to describe this result.

B. Steepest Descent

The evaluation of the integral was also carried out by the method of steepest descent. Along each ray in $X \sim T$ plane the dispersion series was used to find the values of ω and $\partial\omega/\partial\alpha$ as well as the second derivative $\partial^2\omega/\partial\alpha^2$. Having found the appropriate values at the saddlepoint the first term of the expansion could be evaluated from Eq. (6) for the values of X that were used in the summation exercise. The results are quite similar to those shown on Fig. 1 and are compared with the direct summation data on Fig. 2a, plotted against X/T, which is a more appropriate parameter for an asymptotic model.

C. Gaussian Model

The parameters appearing in Eq. (7) define the behavior of the dispersion in the neighborhood of the most unstable wave. These were computed from the series dispersion formulation and used to determine the solution according to the approximation of Eq. (7) for similar values of X and T used previously. The results are compared with the summation solutions on Fig. 2b.

D. Real Axis Model

The series defining ω as a function of α was used to obtain solutions to the equation $X/T = \partial\omega/\partial\alpha$ for a range of X/T values, and these were used in Eq. (11) to give the wave forms in the physical plane. It can be seen for Fig. 3 that there are no solutions for X/T greater than about 0.43 because the real part of $\partial\omega/\partial\alpha$ reaches a maximum at some point along the real α-axis.

IV. Discussion

The numerical evaluation of the integral representing a wave packet has been compared with various approximations. The "saddlepoint" expansion gives the closest agreement with the summation solution (considered in the present context to be "exact"), and even provides a good representation as near to the source as 100 displacement thicknesses (or 5 wavelengths). The solution arises as an asymptotic series in inverse powers of T. The magnitude of the error incurred by only retaining the leading term of the series is of order $1/T$ times the leading term, and thus the error along any ray is relative to the amplitude of the leading term along that ray. The Gaussian model, on the other hand, provides a global solution with errors linked to the maximum disturbance at the center of the packet. At large distances downstream, the Gaussian formulation will provide a very good approximation to the wave packet, but closer to the source the first term of the steepest descent solution is likely to be superior, particularly in resolving the tails of the wave forms.

When attempts were made to calculate the shapes of three-dimensional wave packets some time ago by the author, there were no simple means for solving the saddlepoint equations, and what at the time seemed a reasonable further approximation was made. This in essence was the "real axis method." By expanding the dispersion relations about the real axis it was shown, quite correctly, that this solution was valid if certain terms were small. The assumption was made that the terms were in fact small, without justification as it turned out, and some rather interesting wave packet shapes emerged. In a more complex situation Landahl[3] made a similar approximation. This has been quite properly challenged by Stewartson,[2] who showed that it was only a correct form of approximation when certain conditions on the dispersion relation are met. So although it is now recognized that the approximation is not strictly correct for instability waves in either a boundary layer or in a channel flow, there is still a belief among some that the errors may not be all that great. This stems from the view that the true result cannot be all that different from that given by purely kinematic arguments. Of course the dominant ray, given when Eq. (1) lies on the real axis where ω_i is a maximum, will be correct because $\partial\omega_i/\partial\alpha$ is also zero. Also, for rays very close to the dominant one the solution can be expected to be correct, but only over a very narrow range. Far downstream (of the order of 5000-10,000 displacement thicknesses), therefore, the real axis model will approach a Gaussian form in the central region, but this does not provide a useful description of the behavior of a packet at reasonable distances from the source. For the particular case of the boundary layer on a flat plate, there is no doubt that the numerical evaluations carried out here indicate the true magnitude of the errors incurred by the real axis approach.

V. Conclusions

The computation of wave packet development in the linear regime has been discussed and it has been shown that a good approximation is provided by the first term of the method of steepest descent. The Gaussian expansion will also provide an adequate representation at large distances from the source. But, in this boundary layer example at least, the idea of calculating ray trajectories from the real part of the dispersion is quite unacceptable and can lead to incorrect results.

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